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Crossed modules for Lie–Rinehart algebras

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Abstract

We introduce the notion of crossed module for Lie–Rinehart algebras and prove that they are classified by the third cohomology of Lie–Rinehart algebras developed in [J. Huebschmann, J. Reine Angew. Math. 408 (1990) 57–113; G.S. Rinehart, Trans. Amer. Math. Soc. 108 (1963) 195–222].

Keywords: Lie–Rinehart algebra; Crossed modules; Cohomology

1. Introduction

Lie–Rinehart algebras play an important role in many branches of mathematics, see [3] and references given there. In this paper we introduce the notion of crossed module for Lie–Rinehart algebras, which generalizes the similar notion for Lie algebras introduced by Kassel and Loday [4].

Our main interest is to relate the crossed modules with cohomology of Lie–Rinehart algebras. Our main result claims that the third dimensional cohomology of Lie–Rinehart algebras classifies crossed modules of Lie–Rinehart algebras. This result is in the same spirit as the classical result for group cohomology due to Loday and others.

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2. Preliminaries on Lie–Rinehart algebras

2.1. Motivations, definitions and examples

We start by recalling the definition of Lie–Rinehart algebras. Let K be a field and A be a commutative algebra over K. We let Der(A) be the set of all K-derivations of A. Thus elements of Der(A) are K-linear maps $D:\mathrm{A}\to\mathrm{A}$ such that $D(ab)=aD(b)+D(a)b$ holds. It is well-known that Der(A) is a Lie K-algebra under the bracket

$$\left[D,D'\right]=DD'-D'D.$$

For $a\in\mathrm{A}$ and $D\in\mathrm{Der(A)}$ one has $aD\in\mathrm{Der(A)}$, here aD is defined by $(aD)(b)=aD(b)$, $b\in\mathrm{A}$. Thus Der(A) is also an A-module. It is well-known and it is easy to check that the following holds

$$\left[D,aD'\right]=a\left[D,D'\right]+D(a)D',\quad D,D'\in\mathrm{Der(A)}.$$

In particular, Der(A) is not a Lie A-algebra. The above formula leads to the following definition, which goes back to Herz under the name "pseudo-algèbre de Lie" (see [1]).

Following Huebschmann [3], a *Lie–Rinehart algebra* over A consists of a Lie K-algebra $\mathcal{L}$ together with an A-module structure on $\mathcal{L}$ and a map

$$\alpha:\mathcal{L}\to\mathrm{Der(A)}$$

which is simultaneously a Lie algebra and an A-module homomorphism such that

$$[X,aY]=a[X,Y]+X(a)Y \tag{1}$$

holds. Here $X,Y\in\mathcal{L}$, $a\in\mathrm{A}$ and we write $X(a)$ for $\alpha(X)(a)$.

It is clear that the Lie–Rinehart algebras with $\alpha=0$ are exactly the Lie A-algebras. On the other hand, any commutative K-algebra A defines a Lie–Rinehart algebra with $\mathcal{L}=\mathrm{Der(A)}$.

If $\mathfrak{g}$ is a K-Lie algebra acting on a commutative K-algebra A by means of $\gamma:\mathfrak{g}\to\mathrm{Der(A)}$, then the transformation Lie–Rinehart algebra of ($\mathfrak{g}$, A) is $\mathcal{L}=\mathrm{A}\otimes\mathfrak{g}$ with Lie bracket

$$\left[a\otimes g,a'\otimes g'\right]=aa'\otimes\left[g,g'\right]+a\gamma(g)\left(a'\right)\otimes g'-a\gamma\left(g'\right)(a)\otimes g$$

and action $\alpha:\mathcal{L}\to\mathrm{Der(A)}$, $\alpha(a\otimes g)(a')=a\gamma(g)(a')$. This is the infinitesimal analogue of the transformation groupoid associated to an action of a group on a space.

If $\mathcal{L}$ is a Lie–Rinehart algebra over A, then $\mathcal{L}\rtimes\mathrm{A}$ with Lie bracket $[(X,a),(X',a')]=([X,X'],X(a')-X'(a))$ and action $\widetilde{\alpha}:\mathcal{L}\rtimes\mathrm{A}\to\mathrm{Der(A)}$, $\widetilde{\alpha}(X,a)=\alpha(X)$, is a Lie–Rinehart algebra.

Let $\mathbb{R}$ be the real numbers, $\mathrm{A}=C^{\infty}(\mathrm{M})$ the algebra of smooth functions on a manifold M and $\mathcal{L}$ a Lie–Rinehart algebra over A which is finitely generated projective A-module. Then it follows from Swan's theorem that $\mathcal{L}=C^{\infty}(E)$ is the space of smooth sections of a vector bundle over M. The bundle map $\alpha : E \to T\mathrm{M}$ induces $\alpha : C^\infty(E) \to \mathrm{Der}_{\mathbb{R}}(C^\infty(\mathrm{M})) = C^\infty(T\mathrm{M})$. So we recover Lie algebroids as a particular example of Lie–Rinehart algebras.

If $\mathcal{L}$ and $\mathcal{L}'$ are Lie–Rinehart algebras, then a *Lie–Rinehart homomorphism* $f : \mathcal{L} \to \mathcal{L}'$ is a map, which is simultaneously a Lie K-algebra homomorphism and a homomorphism of A-modules. Furthermore one requires that the diagram

$$\begin{tikzcd}
\mathcal{L} \arrow[dd, "f"'] \arrow[dr, "\alpha"] & \\
& \mathrm{Der}(\mathrm{A}) \\
\mathcal{L}' \arrow[ur, "\alpha'"'] &
\end{tikzcd}$$

commutes. We denote by $\mathcal{LR}(\mathrm{A})$ the category of Lie–Rinehart algebras. As we said one has the full inclusion

$$\mathcal{L}(\mathrm{A}) \subset \mathcal{LR}(\mathrm{A})$$

where $\mathcal{L}(\mathrm{A})$ denotes the category of Lie A-algebras. Let us observe that the kernel of any Lie–Rinehart algebra homomorphism is a Lie A-algebra.

2.2. Actions and semi-direct product

Let $\mathcal{L} \in \mathcal{LR}(\mathrm{A})$ and let R be a Lie A-algebra. We will say that $\mathcal{L}$ *acts on* R if a K-linear map

$$\mathcal{L} \otimes R \to R, \quad (X, r) \mapsto [X, r], \quad X \in \mathcal{L},\ r \in R,$$

is given such that the following identities hold

(1) $[[X, Y], r] = [X, [Y, r]] - [Y, [X, r]]$,
(2) $[X, [r_1, r_2]] = [[X, r_1], r_2] - [[X, r_2], r_1]$,
(3) $[aX, r] = a[X, r]$,
(4) $[X, ar] = a[X, r] + X(a)r$.

Let us observe that (1) and (2) mean that $\mathcal{L}$ acts on R in the category of Lie K-algebras.

If R is an abelian Lie A-algebra and $\mathcal{L} \in \mathcal{LR}(\mathrm{A})$ acts on R then we call R as a *Lie–Rinehart module over* $\mathcal{L}$. Let $(\mathcal{L}, \mathrm{A})$-**mod** be the category of Lie–Rinehart modules over $\mathcal{L}$.

Let us consider a Lie–Rinehart algebra $\mathcal{L}$ and a Lie A-algebra R on which $\mathcal{L}$ acts. Since $\mathcal{L}$ acts on R in the category of Lie K-algebras as well, we can form the semi-direct product

$\mathcal{L} \rtimes R$ in the category of Lie K-algebras, which is $\mathcal{L} \oplus R$ as a vector space, equipped with the following bracket

$$\big[(X,r),\big(X',r'\big)\big] := \big(\big[X,X'\big],\big[r,r'\big] + \big[X,r'\big] - \big[X',r\big]\big).$$

We claim that $\mathcal{L} \rtimes R$ has also a natural Lie–Rinehart algebra structure. Firstly, $\mathcal{L} \rtimes R$ as an A-module is the direct sum of A-modules $\mathcal{L}$ and R. Hence $a(X,r) = (aX, ar)$. Secondly the map

$$\widetilde{\alpha} : \mathcal{L} \rtimes R \to \mathrm{Der}(\mathrm{A})$$

is given by $\widetilde{\alpha}(X,r) := \alpha(X)$. In this way we really get a Lie–Rinehart algebra. Indeed, it is clear that $\widetilde{\alpha}$ is simultaneously an A-module and Lie algebra homomorphism and it is obtained (1)

$$\begin{aligned}\big[(X,r),a\big(X',r'\big)\big] &= \big[(X,r),\big(aX',ar'\big)\big] = \big(\big[X,aX'\big],\big[r,ar'\big] + \big[X,ar'\big] - \big[aX',r\big]\big)\\ &= \big(a\big[X,X'\big] + X(a)X', a\big[r,r'\big] + a\big[X,r'\big] + X(a)r' - a\big[X',r\big]\big)\\ &= a\big(\big[X,X'\big],\big[r,r'\big] + \big[X,r'\big] - \big[X',r\big]\big) + \big(X(a)X', X(a)r'\big)\\ &= a\big[(X,r),\big(X',r'\big)\big] + X(a)\big(X',r'\big).\end{aligned}$$

Thus $\mathcal{L} \rtimes R$ is indeed a Lie–Rinehart algebra.

We now give a construction of "differential operators" on a Lie A-algebra which generalizes the construction given in (2.11) of [3].

Let $\mathcal{R}$ be a Lie A-algebra and let $\mathcal{L}$ be a Lie–Rinehart algebra over A. Let $\mathcal{DO}(\mathrm{A},\mathcal{L},\mathcal{R})$ be the vector space of pairs (d,X), where $d : \mathcal{R} \to \mathcal{R}$ is a K-derivation of a Lie K-algebra $\mathcal{R}$ and $X \in \mathcal{L}$, such that

$$d(ar) = ad(r) + X(a)r$$

holds, for $a \in \mathrm{A}$, $r \in \mathcal{R}$. Then the componentwise operations make $\mathcal{DO}(\mathrm{A},\mathcal{L},\mathcal{R})$ an A-module and a Lie K-algebra. Furthermore the composite

$$\mathcal{DO}(\mathrm{A},\mathcal{L},\mathcal{R}) \xrightarrow{pr} \mathcal{L} \xrightarrow{\alpha} \mathrm{Der}(\mathrm{A})$$

can be used to get a Lie–Rinehart algebra structure on $\mathcal{DO}(\mathrm{A},\mathcal{L},\mathcal{R})$. The case of abelian Lie algebras $\mathcal{R}$ was considered in [3].

It is clear that one has an exact sequence of Lie–Rinehart algebras

$$0 \to \mathrm{Der}_A(\mathcal{R}) \to \mathcal{DO}(\mathrm{A},\mathcal{L},\mathcal{R}) \xrightarrow{p} \mathcal{L}$$

where $\mathrm{Der}_A(\mathcal{R})$ is the Lie A-algebra of all A-derivations of the Lie A-algebra $\mathcal{R}$ and $p(d,X) = X$.

One observes that actions of $\mathcal{L}$ on $\mathcal{R}$ is in 1–1 correspondence with Lie–Rinehart algebras homomorphisms $f : \mathcal{L} \to \mathcal{DO}(\mathrm{A}, \mathcal{L}, \mathcal{R})$ for which the diagram

$$\begin{tikzcd} \mathcal{DO}(\mathrm{A}, \mathcal{L}, \mathcal{R}) \arrow[r, "p"] & \mathcal{L} \\ \mathcal{L} \arrow[u, "f"] \arrow[ur, equal] & \end{tikzcd}$$

commutes. Indeed, having such f, for any $X \in \mathcal{L}$ one has $f(X) = (d_X, X)$. Now

$$[X, r] := d_X(r), \quad X \in \mathcal{L},\ r \in \mathcal{R}$$

defines an action of $\mathcal{L}$ on $\mathcal{R}$ and any action of $\mathcal{L}$ on $\mathcal{R}$ comes in this way.

2.3. *Cohomology of Lie–Rinehart algebras*

Let us recall also the definition of the cohomology $H^{\star}(\mathcal{L}, \mathrm{M})$ of a Lie–Rinehart algebra $\mathcal{L}$ with coefficients in a Lie–Rinehart module M (see [3] and [6]). One puts

$$C^n(\mathcal{L}, \mathrm{M}) := \mathrm{Hom}_{\mathrm{A}}\big(\Lambda^n_{\mathrm{A}}\mathcal{L}, \mathrm{M}\big)$$

where $\Lambda^*_{\mathrm{A}}(V)$ denotes the exterior algebra over A generated by an A-module V. The coboundary map

$$\delta : C^{n-1}(\mathcal{L}, \mathrm{M}) \to C^n(\mathcal{L}, \mathrm{M})$$

is given by

$$\begin{aligned}(\delta f)(X_1, \ldots, X_n) = (-1)^n \sum_{i=1}^{n} (-1)^{(i-1)} X_i\big(f\big(X_1, \ldots, \widehat{X}_i, \ldots, X_n\big)\big) \\ + (-1)^n \sum_{j<k} (-1)^{j+k} f\big([X_i, X_j], X_1, \ldots, \widehat{X}_i, \ldots, \widehat{X}_j, \ldots, X_n\big).\end{aligned}$$

Here $X_1, \ldots, X_n \in \mathcal{L}$, $f \in C^{n-1}(\mathcal{L}, \mathrm{M})$. By the definition $H^{\star}(\mathcal{L}, \mathrm{M})$ is the cohomology of the cochain complex $C^{\star}(\mathcal{L}, \mathrm{M})$.

The cohomology $H^2(\mathcal{L}, \mathrm{M})$, when $\mathcal{L}$ is A-projective, classifies abelian extensions of $\mathcal{L}$ by M thanks to Theorem 2.6 in [3].

3. Crossed modules of Lie–Rinehart algebras

Definition 1. *A crossed module* $\partial : \mathcal{R} \to \mathcal{L}$ *of Lie–Rinehart algebras over* A consists of a Lie–Rinehart algebra $\mathcal{L}$ and a Lie A-algebra $\mathcal{R}$ together with the action of $\mathcal{L}$ on $\mathcal{R}$ and the Lie K-algebra homomorphism ∂ such that the following identities hold:

(1) $\partial([X, r]) = [X, \partial(r)]$,
(2) $[\partial(r'), r] = [r', r]$,
(3) $\partial(ar) = a\partial(r)$,
(4) $\partial(r)(a) = 0$

for all $r \in \mathcal{R}$, $X \in \mathcal{L}$, $a \in$ A.

The first two conditions say that $\partial : \mathcal{R} \to \mathcal{L}$ is a crossed module of Lie K-algebras (see [4]), the condition (3) says that ∂ is a map of A-modules and the condition (4) says that the composition of the following maps is zero

$$\mathcal{R} \xrightarrow{\partial} \mathcal{L} \xrightarrow{\alpha} \mathrm{Der}(\mathrm{A}).$$

Example 2. (i) For any Lie–Rinehart morphism $f : \mathcal{L} \to \mathcal{L}'$, the diagram

$$\ker f \hookrightarrow \mathcal{L}$$

is clearly a crossed module of Lie–Rinehart algebras.

(ii) Let $\mathcal{L}$ be a Lie–Rinehart algebra over A. A Lie–Rinehart subalgebra $\mathcal{N}$ of $\mathcal{L}$ consists of a K-Lie subalgebra $\mathcal{N}$ which is an A-module and $\mathcal{N}$ acts on A via the composition

$$\mathcal{N} \hookrightarrow \mathcal{L} \xrightarrow{\alpha} \mathrm{Der}(\mathrm{A}).$$

It is said that a Lie–Rinehart subalgebra $\mathcal{N}$ of $\mathcal{L}$ is an ideal if $\mathcal{N}$ is an ideal of $\mathcal{L}$ as K-Lie algebra and the composition

$$\mathcal{N} \hookrightarrow \mathcal{L} \xrightarrow{\alpha} \mathrm{Der}(\mathrm{A})$$

is trivial. Then the inclusion map $i : \mathcal{N} \to \mathcal{L}$ is a crossed module where the action of $\mathcal{L}$ on $\mathcal{N}$ is given by the Lie bracket.

(iii) Let $\mathcal{R}$ be a $(\mathcal{L}, \mathrm{A})$-module. Then the morphism $0 : \mathcal{R} \to \mathcal{L}$ is a crossed module.

(iv) Let $\mathcal{L}$ be a Lie–Rinehart algebra. The centre of $\mathcal{L}$ is the ideal

$$Z(\mathcal{L}) = \big\{X \in \mathcal{L}/[X, Y] = 0,\ \forall Y \in \mathcal{L} \text{ and } X(a) = 0,\ \forall a \in \mathrm{A}\big\}.$$

It is clear that $\mathcal{L}$ is an abelian Lie–Rinehart algebra if and only if $Z(\mathcal{L}) = \mathcal{L}$.

Let $\partial : \mathcal{R} \to \mathcal{L}$ be a central epimorphism (i.e., $\ker \partial \subseteq Z(\mathcal{R})$) from a Lie A-algebra $\mathcal{R}$ to a Lie–Rinehart algebra $\mathcal{L}$. Then $\partial : \mathcal{R} \to \mathcal{L}$ is a crossed module where the action from $\mathcal{L}$ to $\mathcal{R}$ is given by $[X, r] = [r', r]$ such that $\partial(r') = X$.

(v) If $\mathcal{L}$ is a Lie–Rinehart algebra over A, then $\partial : \ker \alpha \to \mathcal{D}O(\mathrm{A}, \mathcal{L}, \ker \alpha)$, $\partial(X) = (ad_X, 0)$, is a crossed module, where the action of $\mathcal{D}O(\mathrm{A}, \mathcal{L}, \ker \alpha)$ on $\ker \alpha$ is $[(d, X), r] = d(r)$.

If $\partial : \mathcal{R} \to \mathcal{L}$ is a crossed module of Lie–Rinehart algebras over A, then $\operatorname{im} \partial$ is simultaneously a Lie K-ideal of $\mathcal{L}$ and an A-submodule, therefore $\operatorname{coker} \partial$ has a natural structure of Lie–Rinehart algebra. Furthermore $\ker\partial$ is an abelian A-ideal of $\mathcal{R}$ and the action of $\mathcal{L}$ on R yields the Lie–Rinehart module structure of $\operatorname{coker}\partial$ on $\ker\partial$.

Let $\mathcal{P}$ be a Lie–Rinehart algebra and let M be a Lie–Rinehart module over $\mathcal{P}$. We consider the category $\mathbf{Cross}(\mathcal{P},\mathrm{M})$, whose objects are the exact sequences

$$0 \to \mathrm{M} \to \mathcal{R} \xrightarrow{\partial} \mathcal{L} \xrightarrow{\upsilon} \mathcal{P} \to 0$$

where $\partial : \mathcal{R} \to \mathcal{L}$ is a crossed module of Lie–Rinehart algebras over A and the canonical maps $\operatorname{coker}\partial \to \mathcal{P}$ and $\mathrm{M} \to \ker\partial$ are isomorphisms of Lie–Rinehart algebras and modules respectively. One requires also that the homomorphisms $\mathcal{L} \to \mathcal{P}$ and $\mathcal{R} \to \operatorname{im}\partial$ have A-linear sections.

The morphisms in the category $\mathbf{Cross}(\mathcal{P},\mathrm{M})$ are commutative diagrams

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{M} \arrow[r] \arrow[d, equal] & \mathcal{R} \arrow[r, "\partial"] \arrow[d, "\alpha"] & \mathcal{L} \arrow[r] \arrow[d, "\beta"] & \mathcal{P} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathrm{M} \arrow[r] & \mathcal{R}' \arrow[r, "\partial'"] & \mathcal{L}' \arrow[r] & \mathcal{P}' \arrow[r] & 0
\end{tikzcd}$$

where β is an A-split homomorphism of Lie–Rinehart algebras, α is an A-split morphism of Lie A-algebras and for any $r \in \mathcal{R}$, $X \in \mathcal{L}$ one has

$$\alpha\big([X,r]\big) = \big[\beta(X), \alpha(r)\big].$$

Now we are in the position to formulate our main result:

Theorem 3. *For any Lie–Rinehart algebra $\mathcal{P}$ which is projective as an* A*-module and any Lie–Rinehart module* M *there exists a natural bijection between the classes of the connected components of the category* $\mathbf{Cross}(\mathcal{P},\mathrm{M})$ *and* $H^3(\mathcal{P},\mathrm{M})$.

The result is quite similar to the classical result for the group cohomology proved by the several people including Loday [5] and Huebschmann [2]. Similar result for Lie algebra cohomology was proved by Kassel and Loday [4]. The proof of Theorem 3 is in the same line as the one given in [4]. First we need the relative version of Theorem 3.

Let $\upsilon : \mathcal{L} \to \mathcal{P}$ be a surjective homomorphism of Lie–Rinehart algebras which has an A-linear section. For any Lie–Rinehart $\mathcal{P}$-module M one can define the cochain complex $C^*(\mathcal{P},\mathcal{L},\mathrm{M})$ via the exact sequence

$$0 \to C^*(\mathcal{P},\mathrm{M}) \xrightarrow{\upsilon^*} C^*(\mathcal{L},\mathrm{M}) \xrightarrow{\kappa^*} C^*(\mathcal{P},\mathcal{L},\mathrm{M}) \to 0.$$

The cohomology of the chain complex $C^*(\mathcal{P},\mathcal{L},\mathrm{M})$ is denoted by $H^{*+1}(\mathcal{P},\mathcal{L},\mathrm{M})$.

One denotes by $\mathbf{Cross}(\mathcal{P},\mathcal{L},\mathrm{M})$ the subcategory of $\mathbf{Cross}(\mathcal{P},\mathrm{M})$ whose objects are of the form

$$0 \to \mathrm{M} \to \mathcal{R} \xrightarrow{\partial} \mathcal{L} \xrightarrow{\upsilon} \mathcal{P} \to 0$$

with fixed υ. The morphisms in $\mathbf{Cross}(\mathcal{P},\mathcal{L},\mathrm{M})$ have the following form

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{M} \arrow[r] \arrow[d, equal] & \mathcal{R} \arrow[r, "\partial"] \arrow[d, "\alpha"] & \mathcal{L} \arrow[r, "\upsilon"] \arrow[d, equal] & \mathcal{P} \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & \mathrm{M} \arrow[r] & \mathcal{R}' \arrow[r, "\partial'"] & \mathcal{L} \arrow[r, "\upsilon"] & \mathcal{P} \arrow[r] & 0 \end{tikzcd}$$

Let us observe that $\mathbf{Cross}(\mathcal{P},\mathcal{L},\mathrm{M})$ is a groupoid, that is all morphisms in $\mathbf{Cross}(\mathcal{P},\mathcal{L},\mathrm{M})$ are isomorphisms.

Theorem 4. *For any surjective Lie–Rinehart algebra homomorphism $\upsilon:\mathcal{L}\to\mathcal{P}$, which has an* A*-linear section there exists a natural bijection between the set of connected components of* $\mathbf{Cross}(\mathcal{P},\mathcal{L},\mathrm{M})$ *and* $H^3(\mathcal{P},\mathcal{L},\mathrm{M})$.

The fact that Theorem 4 implies Theorem 3 is quite formal and uses only the fact that $H^n(\mathcal{P},-)$ vanishes on injective Lie–Rinehart modules for $n\geqslant 1$, which was proved by Rinehart [6]. Thus we have only to prove Theorem 4.

Observe that for $\mathrm{A}=K$, Theorems 3 and 4 give Theorems A.3 and A.2, respectively, in [4].

Proof of Theorem 4. Let $\partial:\mathcal{R}\to\mathcal{L}$ be a crossed module for Lie–Rinehart algebras. We set $\mathcal{N}=\operatorname{im}\partial=\ker\upsilon$. By our assumption we can choose A-linear sections $s:\mathcal{P}\to\mathcal{L}$ and $\sigma:\mathcal{N}\to\mathcal{R}$. Thus $\upsilon s=1_{\mathcal{P}}$ and $\partial\sigma=1_N$. One defines $g:\mathcal{P}\otimes\mathcal{P}\to\mathcal{R}$ by $g(X\otimes Y)=\sigma([sX,sY]-s[X,Y])$. We claim that $g(aX,Y)=ag(X,Y)=g(X,aY)$. Indeed, one easily checks that

$$g(aX,Y)-ag(X,Y)=-\sigma\big((sY)(a)sX-s\big(Y(a)X\big)\big).$$

Since υ is a morphism of Lie–Rinehart algebras one has $Za=(\upsilon Z)(a)$ for any $a\in\mathrm{A}$ and $Z\in\mathcal{L}$. Thus $s(Y)(a)=Y(a)$ and the claim is proved. Next one defines $f:\mathcal{L}\otimes\mathcal{L}\to\mathcal{R}$ by

$$f\big(Z,Z'\big)=g\big(\upsilon Z,\upsilon Z'\big)-\big[Z',\psi(Z)\big]+\big[Z,\psi\big(Z'\big)\big]-\big[\psi(Z),\psi\big(Z'\big)\big]+\psi\big(\big[Z,Z'\big]\big)$$

where $\psi:\mathcal{L}\to\mathcal{R}$ is an A-linear map given by

$$\psi(Z):=\sigma\big(Z-s\upsilon(Z)\big).$$

Thanks to [4] f has values in M.

Our second claim is that

$$f\big(aZ,Z'\big)=af\big(Z,Z'\big)=f\big(Z,aZ'\big)$$

for all $Z,Z'\in\mathcal{L}$, $a\in\mathrm{A}$. Indeed, since υ,ψ are A-linear and $g(\upsilon Z,\upsilon Z')$, $[\psi(Z),\psi(Z')]$ are bilinear on Z and Z' we have

$$\begin{aligned} f\big(aZ, Z'\big) - af\big(Z, Z'\big) &= -\big[Z', a\psi(Z)\big] + \big[aZ, \psi\big(Z'\big)\big] \\ &= -\psi\big(\big[aZ, Z'\big]\big) + a\big[Z', \psi(Z)\big] - a\big[Z, \psi\big(Z'\big)\big] + a\psi\big(\big[Z, Z'\big]\big) \\ &= -Z'(a)\psi(Z) - \psi\big(Z'(a)Z\big) = 0. \end{aligned}$$

Similarly $f(Z, aZ') = af(Z, Z')$. Hence $f \in C^2(\mathcal{L}, \mathrm{M})$ and the computation in [4] shows that the image $\kappa^* f$ in $C^2(\mathcal{P}, \mathcal{L}, \mathrm{M})$ is a cocycle whose class in $H^3(\mathcal{P}, \mathcal{L}, \mathrm{M})$ does not depend on the choice of s and σ. Thus one obtains the map

$$\pi_0\big(\mathbf{Cross}(\mathcal{P}, \mathcal{L}, \mathrm{M})\big) \to H^3(\mathcal{P}, \mathcal{L}, \mathrm{M}).$$

To finish the proof of the theorem we have to define the map in the opposite direction

$$H^3(\mathcal{P}, \mathcal{L}, \mathrm{M}) \to \pi_0\big(\mathbf{Cross}(\mathcal{P}, \mathcal{L}, \mathrm{M})\big).$$

Still according to [4], if we have a cocycle in $C^2(\mathcal{P}, \mathcal{L}, \mathrm{M})$ it can lift to a cochain $f \in C^2(\mathcal{L}, \mathrm{M})$. The fact that $\kappa^* f$ is a cocycle in $C^2(\mathcal{P}, \mathcal{L}, \mathrm{M})$ means that $\partial f = \upsilon^* \kappa$, for some $\kappa \in C^3(\mathcal{P}, \mathrm{M})$. Let $\mathcal{R} = \mathcal{N} \oplus \mathrm{M}$ as A-modules. One puts

$$\big[(X, m), \big(X', m'\big)\big] = \big(\big[X, X'\big], f\big(X, X'\big)\big)$$

for all $X, X' \in \mathcal{N}$. Then $\mathcal{R}$ is a Lie A-algebra, because the restriction of f on $\mathcal{N} \times \mathcal{N}$ is a cocycle, thanks to equation $\partial f = \upsilon^* \kappa$. Moreover, by [4] the formula

$$\big[Y, (X, m)\big] = \big([\upsilon Y, m] + f(X, Y), [Y, X]\big),$$

$\partial(X, m) = X \in \mathcal{N}$, $m \in \mathrm{M}$, $Y \in \mathcal{L}$ defines the crossed module $\partial : \mathcal{R} \to \mathcal{L}$ in the category of Lie K-algebras. By definition ∂ is an A-linear map and since $\operatorname{im} \partial = \ker \upsilon$ it follows that $\partial : \mathcal{R} \to \mathcal{L}$ is also a crossed module in $\mathcal{LR}(\mathrm{A})$. □

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